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## *A Geometric Proposition.*

BY E. LASKER.

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The elementary proposition that, “if the corners  $(A, B, C)(D, E, F)$  of two triangles in a plane are such that  $AD, BE, CF$  are collinear, then the reciprocal property is true of the sides respectively opposite to the points,” admits of a very wide extension that may not be without some importance. The fact, which is going to be explained, has applications to the geometry of a space of  $n$  manifoldness; but, to simplify the diction, it will be described as applying only to geometry in a plane or space. It may be announced in this fashion:

*Proposition.* Let  $A, B, C, D, E, F$  be six points in a plane, or  $A, B, C, D, E, F, G, H$  be eight points in space. Let  $\Omega$  be any configuration of these points, which is characterized by the vanishing of one or a set of linear invariants  $i$  of above set of points respectively. In that case, any set of points forming  $\Omega$  gives rise, by separating the points into two triangles or tetrahedra, to six sides or eight planes which will always again form  $\Omega$ .

To be quite accurate, we add that if  $A, B, C, D, E, F$  form  $\Omega$ , and the two triangles, into which the six points are divided, are  $(A, B, C)(D, E, F)$ , then the lines  $BC, CA, AB, DE, EF, FD$  form the reciprocal  $\Omega$ . And similarly in space.

The demonstration of the proposition is thus: An invariant  $i$  linear in the coefficients of  $A, B, C, D, E, F$  is of the shape

$$c_1(ABC) \cdot (DEF) + c_2(ABD) \cdot (CEF) + c_3(ADE) \cdot (BCF) + \dots,$$

where the  $c_1, c_2, c_3 \dots$  are numerical constants. Replacing in above expression  $A$  by  $BC, B$  by  $CA, C$  by  $AB, D$  by  $EF, E$  by  $FD, F$  by  $DE$  and form-

ing the reciprocal invariant  $i'$ , it is found by elementary properties of determinants that

$$i' = (ABC) \cdot (DEF) \cdot i.$$

Hence, if  $i = 0$ , also  $i' = 0$ , and if a set of invariants  $i = 0$ , then also the corresponding set  $i' = 0$ . Q. E. D.

The demonstration of the proposition for higher spaces is analogous.

To give a few easy applications: If eight points  $A, B, C, D, E, F, G, H$  in space are such that the line common to the planes  $ABC$  and  $DEF$  intersects  $GH$ , then will the line common to  $ADE$  and  $BCF$ , the point of intersection of  $DEG$  and  $FH$ , and that of  $BCH$  and  $AG$  lie in one plane. Or else: If the quadric through the lines  $AB, CD, EF$  admits  $G, H$  as conjugate points, then will the quadric through  $CD, AB, GH$  admit  $EFH, EFG$  as conjugate planes; and the quadric through  $CG, DH$  and the line common to  $ABG$  and  $EFH$  will admit  $ABC$  and  $DEF$  as conjugate planes; and the quadric through  $CGH/DEF$  (read: the line common to  $CGH$  and  $DEF$ ),  $AGH/BEF$  and  $BD$  will admit  $ACG$  and  $ACH$  as conjugate planes.

It is, of course, possible to apply the proposition to the linear invariants of  $2m$  elements of a  $m$ -fold linear manifoldness. As an instance, let the manifoldness in question be that of the conics in a given plane whose  $m$  is 6. Let

$$u_1, u_2 \dots u_{12}$$

be twelve conics, and let the vanishing of the invariant  $i$  signify that there is a conic belonging to all three involutions

$$(u_1, u_2, u_3, u_4), (u_5, u_6, u_7, u_8) \text{ and } (u_9, u_{10}, u_{11}, u_{12}).$$

Interpreting the  $u_1 \dots u_{12}$  as squares of lines, the proposition means this: "If three quadruples  $(l_1, l_2, l_3, l_4), (l_5, l_6 \dots)(l_9 \dots)$  of lines are said to be in relation  $\Omega$  whenever three conics exist touching the respective quadruples and having in common four tangents, then any system of lines  $l_1 \dots l_{12}$  in relation  $\Omega$  determines systems of points  $p_1 \dots p_{12}$  also in relation  $\Omega$ . The points  $p_1 \dots p_{12}$  are found in this fashion: Any two quadruples of lines  $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$  and  $(\lambda_5 \dots \lambda_8)$  always uniquely determine a quadruple of points  $(\pi_1, \pi_2, \pi_3, \pi_4)$ , namely, those four points whose squares form the linear involution comprising

the systems of conics that touch the quadruples of lines respectively. Thus,

	$(p_1, p_2, p_3, p_4)$
are determined by	$(l_5, l_6, l_9, l_{10})$
and	$(l_7, l_8, l_{11}, l_{12}),$
$p_5, p_6, p_7, p_8$ by	$(l_9, l_{10}, l_{11}, l_{12})$
and	$(l_{11}, l_{12}, l_3, l_4),$
$p_9, p_{10}, p_{11}, p_{12}$ by	$(l_1, l_2, l_5, l_6)$
and	$(l_3, l_4, l_7, l_8).$

These applications will have given a sufficient insight into the extent as well as the limitations of the proposition.

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